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The stability of natural convection in a vertical fluid layer

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The stability of natural convection in fluid between two parallel vertical plates is investigated theoretically. The two plates are maintained at different temperatures and a uniform stable temperature gradient β is present in the vertical direction. The Prandtl number of the fluid is fixed at 7·5. An orthonormalization method is used in numerical integrations of the disturbance equations. It is shown how the critical Grashof number varies with β for both stationary and travelling disturbances. It is found that for $\beta \leqslant 7{\cdot}1 \times 10^{-3}$ the convection is unstable to stationary disturbances and for $\beta \geqslant 7{\cdot}1 \times 10^{-3}$ it is unstable to travelling disturbances. The critical Grashof number is given by

$$G_c = \begin{cases} 500 & \text{for} \quad \beta < 1{\cdot}0 \times 10^{-3}, \\ 1{\cdot}3 \times 10^6 \beta^3 & \text{for} \quad \beta > 4{\cdot}1 \times 10^{-2}, \end{cases}$$

and even for intermediate values of β the variation of G_c is rather simple but not monotonic.

1. Introduction

In fluid filling the space between two parallel vertical plates, natural convection occurs when the plates are maintained at different temperatures. The convection intensifies with increasing temperature difference $2\Delta T^*$ and becomes unstable when the difference exceeds some critical value. This paper contains a theoretical study of the stability of a temperature and flow field established as a result of natural convection.

In the limiting case of an infinitely deep fluid, it has been established that convection is unstable to stationary disturbances if the Prandtl number of the fluid is less than 10 (Gershuni 1953; Rudakov 1967; Vest & Arpaci 1969; Gotoh & Ikeda 1972). This theoretical result has been confirmed by experiments: a stationary disturbance is observed to grow, producing an unstable state of convection, in an air layer with an aspect ratio $h = H^*/2L^* = 33{\cdot}3$, where H^* and L^* are respectively the depth and half-width of the fluid layer (Vest & Arpaci 1969).

In another limiting case, $h = 0$, the problem is reduced to the stability of natural convection along a single vertical plate. A theoretical investigation of this problem was made by Nachtsheim (1963), who found that the convection was unstable to travelling disturbances. This prediction, as well as the curve of neutral stability, has been confirmed experimentally by Polymeropoulos & Gebhart (1967). Convection in a water layer with $h = 16{\cdot}3$ has also been found to be unstable to travelling disturbances (Oshima 1971).

A change in the mode of instability can be expected, therefore, at some finite value of h, h_c say. An objective of the present investigation is to confirm the existence of h_c and to find its value in the case of a water layer. Heat transfer across the fluid layer depends sensitively on the mode of fluid motion, so the location of h_c is of importance in such a problem as 'double glazing', for example (Batchelor 1954).

It has been found experimentally that the effect of the fluid layer being of finite depth appears essentially as a temperature gradient in the vertical direction, the magnitude of which is equal to $\Delta T^*/(2hL^*)$. So, instead of studying the stability of the exact laminar flow solution in a fluid layer with finite aspect ratio, we examine natural convection driven in a fluid layer between two *infinite* vertical plates in the presence of a uniform stable temperature gradient β in the vertical direction throughout the system. The velocity and the temperature distributions in this problem give a reasonable fit to observations made near the centre-line of a fluid layer with large aspect ratio h (Elder 1965).

An orthonormalization method is used in numerical integrations of the disturbance equations. The β dependence of the critical Grashof number will be found for each of two modes of instability. From this it will be concluded that if $\beta > 7{\cdot}1 \times 10^{-3}$ the convection is unstable to travelling (two-dimensional) disturbances and that otherwise it is unstable to stationary (two-dimensional) disturbances, i.e. $\beta_c = 7{\cdot}1 \times 10^{-3}$. This is rewritten as $h_c = 70{\cdot}6$ using the empirical relation $\beta = (2h)^{-1}$.

2. Formulation of the problem

We consider a fluid layer between two parallel vertical plates at $x^* = \pm L^*$, where the x^* axis of the Cartesian co-ordinates is taken perpendicular to the plates and the z^* axis vertically upwards. The plate at $x^* = -L^*$ is maintained at a temperature $T^* = T_1^* + \beta^* z^*$ and the other at $T^* = T_2^* + \beta^* z^*$, where T_1^* and T_2^* are constants. If we restrict our interest to the two-dimensional motion and the temperature distribution, the fundamental equations may be written, under the Boussinesq approximation, as follows:

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial z}\right)\Delta\psi = \frac{1}{G}\left(\Delta^2\psi - \frac{\partial T}{\partial x}\right), \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial z}\right)T = \frac{1}{\sigma G}\Delta T, \tag{2.2}$$

where $\psi(x, z;\, t)$ is the stream function and $T(x, z; t)$ the temperature relative to $\frac{1}{2}(T_1^* + T_2^*)$. All quantities have been made non-dimensional using L^*, ΔT^*

$[=\frac{1}{2}(T_2^*-T_1^*)]$ and $W_0^*\,(=\gamma^* g^* L^{*2}\Delta T^*/\nu^*)$ as the representative length, temperature and velocity respectively, and the non-dimensional parameters

$$G = W_0^* L^*/\nu^*, \quad \sigma = \nu^*/\kappa^* \tag{2.3}$$

are the Grashof number and the Prandtl number respectively. Here γ^*, ν^* and κ^* are the coefficients of volume expansion, kinematic viscosity and thermal diffusivity of the fluid and g^* is the acceleration due to gravity.

The boundary conditions are

$$\psi(\pm 1, z; t) = \text{constant}, \tag{2.4}$$

$$T(\pm 1, z; t) = \pm 1 + \beta z. \tag{2.5}$$

If we assume the undisturbed velocity and temperature distributions to be $\psi = \Psi(x)$ and $T = \Theta(x) + \beta z$, the exact solution of (2.1)–(2.5) can be obtained as follows:

$$W(x) = -\partial\Psi/\partial x = -(2m^2)^{-1}\,\text{Im}\,[f(x,m)], \tag{2.6}$$

$$\Theta(x) = \text{Re}\,[f(x,m)], \tag{2.7}$$

where Re and Im denote respectively real and imaginary parts,

$$f(x,m) = \sinh[(1+i)\,mx]/\sinh[(1+i)\,m] \tag{2.8}$$

and the parameter m is defined by

$$4m^4 = \beta\sigma G. \tag{2.9}$$

For $m \ll 1$, $W(x)$ and $\Theta(x)$ are reduced to

$$W(x) = \tfrac{1}{6}x(1-x^2), \quad \Theta(x) = x. \qquad (2.10), (2.11)$$

In the limit $m \to \infty$, the profiles (2.6) and (2.7) of $W(x)$ and $\Theta(x)$ become of boundary-layer type. In the region near $x = 1$, $W(x)$ and $\Theta(x)$ are expressed in terms of a stretched variable $\xi = m(1-x)$ as

$$W(\xi) = (2m^2)^{-1} e^{-\xi}\sin\xi, \quad \Theta(\xi) = e^{-\xi}\cos\xi. \qquad (2.12), (2.13)$$

To simplify the analysis, only two-dimensional disturbances are studied in this paper even though Squire's theorem does not apply, owing to the presence of the basic vertical temperature gradient ($\beta \neq 0$). Since the undisturbed field is independent of z, if we decompose the disturbances $\hat{\psi}(x,z;t)$ and $\hat{\theta}(x,z;t)$ into harmonic components according to

$$\begin{pmatrix}\hat{\psi}(x,z;t)\\ \hat{\theta}(x,z;t)\end{pmatrix} = \sum_{\alpha}\begin{pmatrix}\phi(x)\\ \theta(x)\end{pmatrix}\exp[i\alpha(z-ct)], \tag{2.14}$$

then each component can be treated separately. The real parameter α (>0) is the wavenumber in the z direction while c_r ($=\text{Re}\,(c)$) denotes the phase velocity and αc_i ($=\alpha\,\text{Im}\,(c)$) the amplification rate of the component. According as c_i is positive, zero or negative, the disturbance is amplified, neutral or damped out. Substituting

$$\left.\begin{aligned} \psi &= \Psi(x) + \phi(x)\exp[i\alpha(z-ct)],\\ T &= \Theta(x) + \beta z + \theta(x)\exp[i\alpha(z-ct)] \end{aligned}\right\} \tag{2.15}$$

into (2.1) and (2.2) and neglecting products of the harmonics, we have the following equations for $\phi(x)$ and $\theta(x)$:

$$(W-c)(\phi''-\alpha^2\phi)-W''\phi = (i\alpha G)^{-1}(\phi^{\text{iv}}-2\alpha^2\phi''+\alpha^4\phi-\theta'), \tag{2.16}$$

$$(W-c)\theta+\Theta'\phi = (i\sigma\alpha G)^{-1}(\theta''-\alpha^2\theta+4m^4\phi'), \tag{2.17}$$

where a prime stands for differentiation with respect to x. The boundary conditions are reduced from (2.4) and (2.5) to

$$\phi(\pm 1) = \phi'(\pm 1) = \theta(\pm 1) = 0. \tag{2.18}$$

In order that (2.16)–(2.18) have non-trivial solutions, an appropriate eigenvalue equation must be satisfied by c, α, G, m and σ. When m is zero or small the flow is unstable to stationary disturbances, characterized by $c_r = 0$, and when m is larger the flow is unstable to travelling disturbances, with non-zero c_r. In fact the eigenvalue problem is divided into two cases: $c_r = 0$ and $c_r \neq 0$. The eigenvalue equation for the neutral ($c_i = 0$) stationary disturbances ($c_r = 0$) may be expressed as

$$G_1 = F_1(\alpha, m, \sigma). \tag{2.19}$$

The critical value G_{1c} is defined by

$$G_{1c}(m,\sigma) = \min_{\alpha} F_1(\alpha, m, \sigma). \tag{2.20}$$

In the same manner the eigenvalue equation for the neutral ($c_i = 0$) travelling disturbances can be written as

$$G_2 = F_2(\alpha, m, \sigma), \quad c_r = F_3(\alpha, m, \sigma) \neq 0, \tag{2.21}$$

and the critical value G_{2c} is defined by

$$G_{2c}(m,\sigma) = \min_{\alpha} F_2(\alpha, m, \sigma). \tag{2.22}$$

The critical Grashof number of the system (2.16)–(2.18) is the smaller of G_{1c} and G_{2c}. Which of G_{1c} and G_{2c} is smaller depends on the values of m and σ.

The asymptotic form of $G_{kc}(m,\sigma)$ ($k = 1, 2$) in the limit $m \to \infty$ will be found as follows. If we make use of new variables

$$\hat{\theta}(\xi) = \theta(\xi)/m^3, \quad \hat{c} = 2m^2 c, \quad \hat{\alpha} = \alpha/m, \quad \hat{G} = G/(2m^3), \tag{2.23}$$

then the equations governing $\hat{\theta}(\xi)$ and $\hat{\phi}(\xi)$ are, in the limit $m \to \infty$,

$$(\hat{W}-\hat{c})(d^2/d\xi^2-\hat{\alpha}^2)\phi-(d^2\hat{W}/d\xi^2)\phi = (i\hat{\alpha}\hat{G})^{-1}\{(d^2/d\xi^2-\hat{\alpha}^2)^2\phi-d\hat{\theta}/d\xi\}, \tag{2.24}$$

$$(\hat{W}-\hat{c})\hat{\theta}+(d\hat{\Theta}/d\xi)\phi = (i\sigma\hat{\alpha}\hat{G})^{-1}\{(d^2/d\xi^2-\hat{\alpha}^2)\hat{\theta}+4d\phi/d\xi\}, \tag{2.25}$$

where $$\hat{W} = e^{-\xi}\sin\xi, \quad \hat{\Theta} = 2e^{-\xi}\cos\xi.$$

The boundary conditions for $\hat{\phi}(\xi)$ and $\hat{\theta}(\xi)$ are reduced from (2.18) to

$$\phi(0) = \phi(\infty) = d\phi(0)/d\xi = d\phi(\infty)/d\xi = \hat{\theta}(0) = \hat{\theta}(\infty) = 0. \tag{2.26}$$

Let us denote by $\hat{G}_{kc}(\sigma)$ the critical value of $\hat{G}$, which will be determined by solving (2.24)–(2.26). Then the asymptotic form of $G_{kc}(m,\sigma)$ may be expressed as

$$G_{kc}(m,\sigma) = 2\hat{G}_{kc}(\sigma)\, m^3 \quad \text{as} \quad m \to \infty. \tag{2.27}$$

3. Solutions of the eigenvalue problem

In this paper we deal with only the flow of water, and the Prandtl number is fixed at $\sigma = 7{\cdot}5$ ($= \sigma_w$, say).

The m dependence of $G_{1c}(m, \sigma_w)$

The eigenvalue problem with respect to stationary disturbances has already been investigated by the authors and the m dependence of $G_{1c}(m, \sigma_w)$ found (Gotoh & Mizushima 1973), so only the result is reproduced here, in figure 1. It will be discussed in connexion with $G_{2c}(m, \sigma_w)$ in §4.

The m dependence of $G_{2c}(m, \sigma_w)$

The case $m = 0$ of the problem has been attacked repeatedly and the critical value $G_{2c}(0, \sigma)$ is given by

$$G_{2c}(0, \sigma) = 3 \times 10^7 \tag{3.1}$$

(Gotoh & Satoh 1966; Gotoh & Ikeda 1971). Rudakov examined the stability characteristics of the disturbances at lower values of G. He found $G_{1c}(0, \sigma) = 500$, but could not obtain any result revising (3.1).

The first step in the present investigation is to obtain the branch of $G_{2c}(m, \sigma_w)$ specified by (3.1). Using conventional asymptotic analysis we found the result depicted in figure 1 (for details of the calculation procedure, refer to Gotoh & Ikeda 1971). The curve in figure 1 has the asymptote

$$G_{2c}(m, \sigma_w) = 3 \times 10^7 m^3 \tag{3.2}$$

for large values of m, which means that

$$\hat{G}_{2c} = 1{\cdot}5 \times 10^7. \tag{3.3}$$

For $m = \infty$, it has been found by Gill & Davey (1969) that there are two different modes of instability to travelling disturbances. They termed one mechanically driven instability (MDI) and the other buoyancy-driven instability (BDI). MDI and BDI are characterized by the mechanism of the energy supply for the growth of disturbances. In MDI energy is supplied by the velocity field of the undisturbed flow, while in BDI it is supplied by the undisturbed temperature field. Unfortunately, in their paper there is no result for G_{2c} for MDI for $\sigma = \sigma_w$ with which (3.3) may be compared. The result (3.3) must, however, correspond to this because no direct effect of the temperature field has been included in the asymptotic analysis used in the derivation of (3.3).†

Another asymptote of $G_{2c}(m, \sigma_w)$, which is one for BDI, is given by

$$G_{2c}(m, \sigma_w) = 21 m^3, \tag{3.4}$$

where the numerical coefficient is decided from the result of Gill & Davey.

† Additional evidence is provided by the eigenvalue of c_r. BDI is characterized by a magnitude of c_r larger than $\max_x [\hat{W}(x)]$, while the c_r corresponding to (3.3) is not more than about 16% of $\max_x [\hat{W}(x)]$.

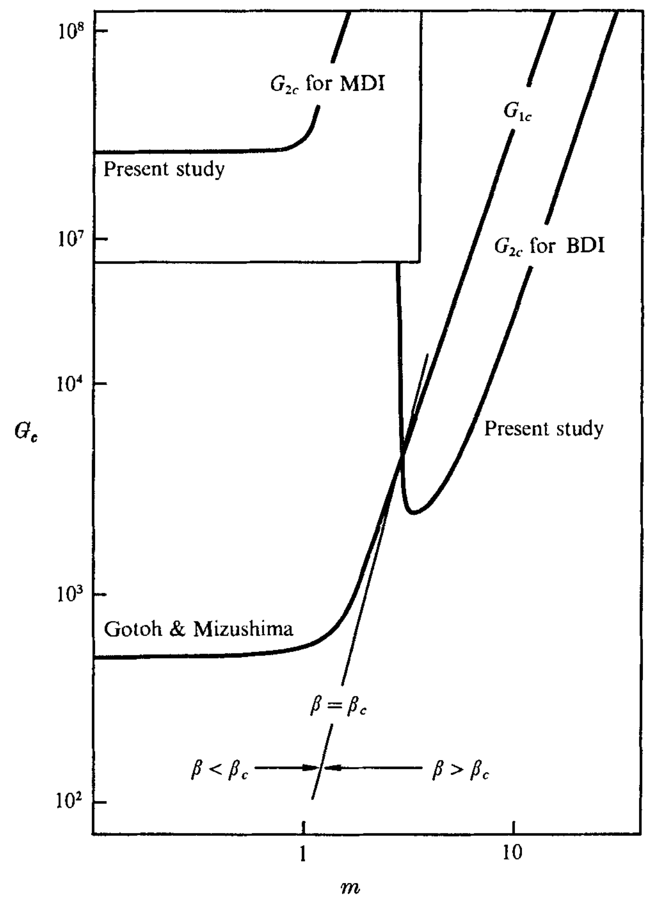

FIGURE 1. The variation in $G_{1c}(m, \sigma)$ and $G_{2c}(m, \sigma)$ with m for $\sigma = 7{\cdot}5$. The critical Grashof number is given for each m by the lowest curve at that m. According to (2.9), β is fixed on straight lines with slope 4. For $\beta < \beta_c$, G_c corresponds to stationary disturbances, while for $\beta > \beta_c$, G_c corresponds to travelling disturbances.

Evidently the $G_{2c}(m, \sigma_w)$ in (3.4) is smaller than that in (3.2), so the critical Grashof number, when $m \gg 1$, is given by the $G_{2c}(m, \sigma_w)$ in (3.4). The second task in this paper is, therefore, to find $G_{2c}(m, \sigma_w)$ for BDI for finite values of m. Since we know no counterpart of G_{2c} for BDI at $m = 0$, we have to use (3.4) as the first approximation in this work. Worse still, the curve of neutral stability for BDI has no asymptote for $\alpha G \to \infty$,† so the asymptotic theory established so far is not applicable and the only way to solve the problem is by numerical integration of the disturbance equations (2.16) and (2.17).

Using a pair of solutions $\phi(x)$ and $\theta(x)$ of (2.16) and (2.17), a complex-valued vector function $\mathbf{a}(x)$ is defined as follows:

$$\mathbf{a}(x) = [\phi(x), \quad \phi'(x), \quad \phi''(x), \quad \phi'''(x), \quad \theta(x), \quad \theta'(x)]. \tag{3.5}$$

† This can be deduced from a study of the Rayleigh equation with the basic flow $W(x)$, details of which are omitted here.

Let us denote by $\mathbf{a}_k(x)$ $(k = 1, 2, 3)$ the functions $\mathbf{a}(x)$ whose values are specified at $x = -1$ as follows:

$$\left.\begin{aligned} \mathbf{a}_1(-1) &= [0, 0, 1, 0, 0, 0], \\ \mathbf{a}_2(-1) &= [0, 0, 0, 1, 0, 0] \\ \mathbf{a}_3(-1) &= [0, 0, 0, 0, 0, 1]. \end{aligned}\right\} \tag{3.6}$$

Then the general solution which satisfies the boundary conditions at $x = -1$ is given by

$$\mathbf{a}(x) = \sum_{k=1}^{3} A_k \mathbf{a}_k(x), \tag{3.7}$$

where the A's are constants to be determined so as to satisfy the boundary conditions at $x = 1$. For all the A's not to vanish, the following secular equation must be satisfied:

$$E(\alpha, G, c_r, m) = \begin{vmatrix} \phi_1(1) & \phi_2(1) & \phi_3(1) \\ \phi_1'(1) & \phi_2'(1) & \phi_3'(1) \\ \theta_1(1) & \theta_2(1) & \theta_3(1) \end{vmatrix} = 0, \tag{3.8}$$

which is an eigenvalue equation for α, G, c_r and m.

The elements of the determinant in (3.8) are evaluated by integrating (2.16) and (2.17) from $x = -1$ to $x = 1$. For large values of αG the particular solutions $\mathbf{a}_k(x)$, as is well known, lose their mutual independence in the process of the numerical integration. Luckily this difficulty may be overcome by making use of the technique of orthonormalization of solutions (Betchov & Criminale 1967, p. 275). After integration of (2.16) and (2.17) over several steps (to $x = x_1$, say) the $\mathbf{a}_k(x_1)$ are made mutually orthogonal, as well as normalized, by replacing $\mathbf{a}_k(x_1)$ by $\mathbf{b}_k(x_1)$ $(k = 1, 2, 3)$, where

$$\left.\begin{aligned} \mathbf{b}_1(x_1) &= N_1 \mathbf{a}_1(x_1), \\ \mathbf{b}_2(x_1) &= N_2\{\mathbf{a}_2(x_1) + B_1 \mathbf{b}_1(x_1)\}, \\ \mathbf{b}_3(x_1) &= N_3\{\mathbf{a}_3(x_1) + B_2 \mathbf{b}_2(x_1) + B_3 \mathbf{b}_1(x_1)\}. \end{aligned}\right\} \tag{3.9}$$

Here the N's are normalization constants and the B's are coefficients determined so as to make

$$[\bar{\mathbf{b}}_1, \mathbf{b}_2] = [\bar{\mathbf{b}}_1, \mathbf{b}_3] = [\bar{\mathbf{b}}_2, \mathbf{b}_3] = 0, \tag{3.10}$$

where $[\mathbf{b}_j, \mathbf{b}_k]$ denotes the inner product of the vectors $\mathbf{b}_j$ and $\mathbf{b}_k$ and a bar a complex conjugate. Integration is continued from $x = x_1$ with the initial values $\mathbf{b}_k(x_1)$ $(k = 1, 2, 3)$. The procedure of orthonormalization is repeated many times, if necessary, and eventually we have the elements of the determinant in (3.8) in orthonormalized form.

We solve the eigenvalue equation (3.8) by the following procedure. Find first a root c_r of Re $[E(\alpha, G, c_r, m)] = 0$ for given values of m, G and α, then repeat the calculation for different values of α and plot the results on the α, c_r plane. Interpolation of the points plotted on the α, c_r plane gives a contour Γ_r on which Re $(E) = 0$. In the same manner a contour Γ_i on the α, c_r plane on which $\mathrm{Im}(E) = 0$ is obtained. Then find an intersection of two contours Γ_r and Γ_i; this gives the eigenvalues α and c_r for given values of m and G. The eigenvalues α and c_r obtained for different values of G give a curve of neutral stability on the α, αG plane for a fixed value of m.

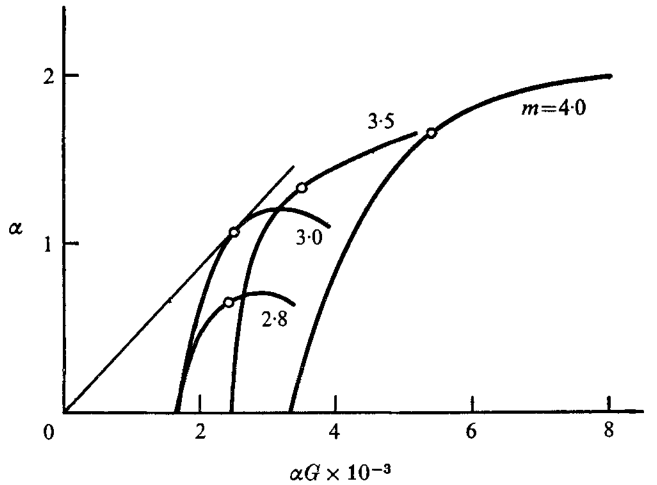

FIGURE 2. Neutral curves for buoyancy-driven instability for different values of m. ○, critical values listed in table 1.

m	G_c	α_c
4·0	$3{\cdot}4 \times 10^3$	1·6
3·5	$2{\cdot}6 \times 10^3$	1·4
3·0	$2{\cdot}4 \times 10^3$	1·1
2·8	$3{\cdot}5 \times 10^3$	0·7

TABLE 1

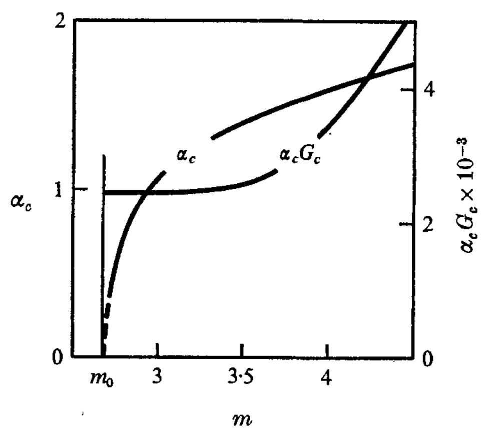

FIGURE 3. Values of m and $\alpha_c G_c$ in the limit $\alpha_c \to 0$.

The curves of neutral stability thus obtained for $m = 4{\cdot}0$, $3{\cdot}5$, $3{\cdot}0$ and $2{\cdot}8$ are plotted in figure 2. The values of $G_{2c}(m, \sigma_w)$ which have been produced from data in figure 2 are given in table 1 together with α_c, the corresponding value of α, and plotted in figure 1. For $m = 2{\cdot}6$ no eigenvalue for neutral stability is found in the whole range of parameters covered by the numerical calculation. From the result in figure 3 it can be conjectured that α_c and $\alpha_c G_c$ approach 0 and $2{\cdot}5 \times 10^3$ respectively as m tends to a certain value between $2{\cdot}6$ and $2{\cdot}8$, m_0 say, so that $G_c \to \infty$ as $m \to m_0$. This conjecture is consistent with the fact that we have no counterpart of $G_{2c}(m, \sigma_w)$ for BDI at $m = 0$.

4. Conclusions

Let us denote by m_c the value of m at which the curves of G_{1c} and G_{2c} for BDI intersect in figure 1. The results presented in figure 1 prove that

$$m_c = 2{\cdot}7 \tag{4.1}$$

and

$$\left.\begin{array}{l} G_{1c} \leqslant G_{2c} \text{ for BDI} \quad \text{and} \quad G_{2c} \text{ for MDI} \quad \text{for} \quad m \leqslant m_c, \\ G_{2c} \text{ for BDI} < G_{1c} < G_{2c} \text{ for MDI} \quad \text{for} \quad m > m_c, \end{array}\right\} \tag{4.2}$$

from which we conclude that the critical Grashof number G_c is given by

$$G_c = \begin{cases} G_{1c} & \text{for} \quad m \leqslant m_c, \\ G_{2c} \text{ for BDI} & \text{for} \quad m > m_c. \end{cases} \tag{4.3}$$

The undisturbed field is, therefore, unstable to stationary disturbances when $m < 2{\cdot}7$ and unstable to travelling disturbances when $m > 2{\cdot}7$. In other words, the mode of growing disturbance is quite different according as $m \lesseqgtr 2{\cdot}7$.

The condition $m \lesseqgtr m_c$ may be represented in terms of β as

$$\beta \lesseqgtr \beta_c, \tag{4.4}$$

where
$$\beta_c = 4m_c^4/\sigma G_{1c}(m_c, \sigma), \tag{4.5}$$

or in terms of h as
$$h \gtreqless h_c \quad (=(2\beta_c)^{-1}), \tag{4.6}$$

so long as the empirical relation $\beta = (2h)^{-1}$ is applicable. For $m_c = 2{\cdot}7$, $G_{1c}(m_c, \sigma_w) = 4{\cdot}0 \times 10^3$ and $\sigma_w = 7{\cdot}5$, the values of β_c and h_c are given respectively by

$$\beta_c = 7{\cdot}1 \times 10^{-3}, \quad h_c = 70{\cdot}6. \tag{4.7}$$

Now the results should be discussed in comparison with those for large values of σ. The results for $\sigma = 1000$ from the paper of Gill & Kirkham (1970) are reproduced in figure 4, and show that the critical Grashof number is given by

$$G_c = \begin{cases} G_{2c} \text{ for BDI} & \text{for} \quad m < m_c', \\ G_{1c} & \text{for} \quad m > m_c', \end{cases} \tag{4.8}$$

where m_c' is the value of m at the intersection of G_{1c} and G_{2c}. This result is quite different from the present one in figure 1 or in (4.3), and there must be some interesting transition as σ increases with respect to preference for stationary as

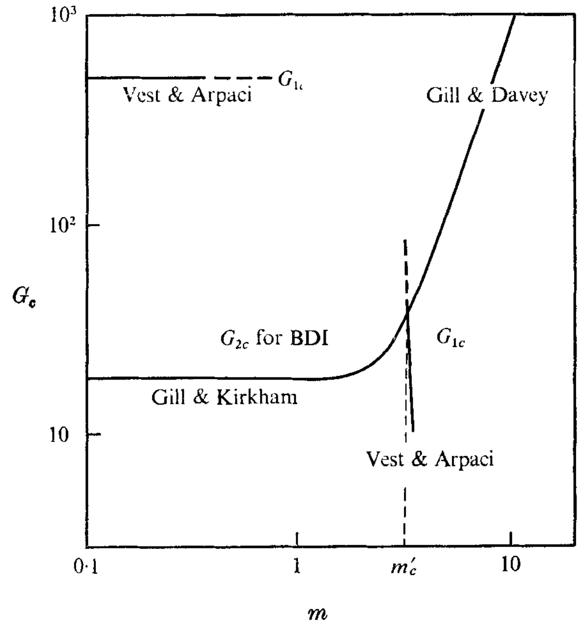

FIGURE 4. Variation of $G_{1c}(m, \sigma)$ and $G_{2c}(m, \sigma)$ with m for $\sigma = 1000$.

opposed to travelling disturbances at large values of m. However, the result for large m of Vest & Arpaci is presumably erroneous, as pointed out by Gill & Kirkham, because Vest & Arpaci ignored the effect of the vertical temperature gradient in the disturbance equations. A future investigation will try to obtain the correct dependence of G_{1c} on m when $\sigma \gg 1$, and a conclusive discussion on the critical Grashof number must be postponed until the end of the programme.

If the discussion is restricted to the transition curve for travelling disturbances, the present result to be used for comparison is the curve G_{2c} in figure 1, which consists of G_{2c} for MDI for $m < m_a$ and G_{2c} for BDI for $m > m_a$, where m_a is the value of m at the intersection of G_{2c} for BDI and G_{2c} for MDI in figure 1. In figure 4, on the other hand, G_{2c} is given by G_{2c} for BDI for any value of m. So some interesting change in the structure of the transition curve G_{2c} for travelling disturbances must occur as σ increases.

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